Abstract algebra season 2

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1. Nakayama's lemma. Let M be a finitely generated A-module and I an ideal of A contained in the Jacobson radical $= \cap M$, M maximal ideal. Prove: $IM = M \implies M = 0$.
Let $\{m_i\}_I$ a minimal generatoring set of M. Given that $IM = M$ we have that $m_1 = \sum a_i m_i$ for certain $a_i \in I$. Then $m_1(1 - a_1) = \sum_{i>1} a_i m_i$. Now if $1 - a_1$ is invertible we would have that m_1 is a linear combination of the other elements of generating set, a contradiction with the minimality. Indeed $a + 1$ where a is in the Jacobson radical is invertible. First notice that if $a + 1$ is in any of the maximal ideals then $1 + a - a = 1$ would also be what implies that the ideal would be the total. Then $(a + 1) = I$ so exists an element λ s.t. $\lambda(a + 1) = 1$ and we are finished.

2. Under the previous hypothesis, prove:
(i) $A/I \otimes_A M = 0 \implies M = 0$
From problem 11 we know that $A/I \otimes_A M \simeq M/IM$ then we have that $M/IM = 0 \implies M = IM$ and by 1 we have $M = 0$.
(ii) If $N \subset M$ is a submodule, $M = IM + N \implies M = N$.
We have that $M/N = IM/N + N/N = IM/N + \overline{0} = IM/N$ so $(M/N) = I(M/N)$ and by 1 $M/N = 0 \implies M = N$.
(iii) If $f : N \to M$ is a homomorphism, $\overline{f} : N/IN \to M/IM$ surjective $\implies f$ surjective.
$f(IN) \subseteq IM$ given that $f(an) = af(n)$ where a is in the ideal and $n \in N$ and $f(n) \in M$. Now $\overline{f}(N/IN) = M/IM = f(N)/IM$. So $(f(N) - M)/IM = 0 \implies f(N) - M \subseteq IM \implies f(N) + M = IM$ so by ii) (given that $f(N) \subseteq M$) $f(N) = M$ and we are finished.

3. Let (A, m) be a local ring and M be a finitely generated A-module, $x_1, ..., x_n$ elements of M. Using Nakayama's lemma prove that:
(i) $x_1, ..., x_n$ generate M over $A \iff \overline{x}_1, ..., \overline{x}_n$ generate M/m over A/m.
First we have to notice that $M/\mathrm{m}M$ over A is isomorphic to $M/\mathrm{m}M$ over A/m. This is, multiplying by a or by $\overline{a}$ is the same operation. If we take $\overline{m} \in M/\mathrm{m}M$ we have to check that multiplying by $a \in A$ or by $a + n$ with $n \in \mathrm{m}$ is the same. Indeed $\overline{m}a = \overline{m}(a + n) \iff \overline{m}a - \overline{m}(a + n) = \overline{0} = -\overline{m}n = -\overline{nm} \iff nm \in \mathrm{m}M$ that is true since $n \in \mathrm{m}$
$\implies$ indeed if we have $\overline{x} \in M/\mathrm{m}M$ we know that $x = \sum a_i x_i$ so $\overline{x} = \sum a_i \overline{x}_i$. Then every element of $\mathrm{m}M$ can be expressed as a linear combination of $\overline{x}_1, ..., \overline{x}_n$ and those are generators.
$\impliedby$ Since $\overline{x}_1, ..., \overline{x}_n$ generate $M/\mathrm{m}M$ we have that any $\overline{x} \in M/\mathrm{m}M$ can be expressed as $\overline{x} = \sum a_i \overline{x}_i \iff x = \sum a_i x_i + n$ with $n \in \mathrm{m}M$. From this we can say that $M = \langle x_1, ..., x_n \rangle + \mathrm{m}M$. Now since m is the only maximal ideal of A we can use **2 (ii)** so $\langle x_1, ..., x_n \rangle = M$ and thus we have seen that $x_1, ..., x_n$ generate M.
(ii) $x_1, ..., xn$ is a minimal system of generators of $M \iff \overline{x}_1, ..., \overline{x}_n$ is a basis of the A/m-vector space M/mM.
$\implies$ Suppose that $\overline{x}_1, ..., \overline{x}_n$ wasn't a basis hence without losing generality we cap suppose that $\overline{x_1}, ..., \overline{x}_{n-1}$ also generate $M/\mathrm{m}M$. Then by **(i)** $x_1, ..., x_{n-1}$ would be generators of M contradicting the minimality of $x_1, ..., x_n$.
$\impliedby$ Suppose that $x_1, ..., x_n$ were not minimal hence without losing generality we cap suppose that

$x_1, ..., x_{n-1}$ also generate M. Then by **(i)** $\overline{x_1}, ..., \overline{x_{n-1}}$ would be generators of M contradicting the minimality (given by the fact that they are a basis of a vector space) of $\overline{x_1}, ..., \overline{x}_{n-1}$.

(iii) All minimal systems of generators of M have the same number of elements, equal to the dimension of the A/m-vector space M/mM.

We have just seen that a minimal system of generator has to have the same number of elements of a basis of M/mM. In a finite-dimension vectorial space (this one is because there exists a sistem of generators of finite length) all basis have the same number of elements, the dimension of the vector space.

(iv) $x_1, ..., x_n$ are part of a minimal system of $M \iff x_1, ..., x_n$ are linearly independent in M/mM.

$\implies$ Extend $x_1, ..., x_n$ to $x_1, ..., x_n, x_{n+1}, ..., x_m$ the minimal system. Then $\overline{x}_1, ..., \overline{x}_n, \overline{x}_{n+1}, ..., \overline{x}_m$ is a basis and any subset of a basis has to be linearly independent. If it wasn't the basis wouldn't be linear independent contradicting the fact that it is a basis.

$\impliedby$ Any set of linear independent vectors in a finite dimension vector space can be extended into a basis. If $\overline{x}_1, ..., \overline{x}_n, \overline{x}_{n+1}, ..., \overline{x}_m$ was the extension, $x_1, ..., x_n, x_{n+1}, ..., x_m$ would be a minimal set of generators so we are finished.

4. Let A be a non-local ring. Prove that the A-module A has two minimal systems of generators with a different number of generators.

Obviously $\{1\}$ is a minimal generator of cardinal 1. Let I, J two different maximal ideals and let $b \notin I, \in J$ and $\{a\} \subseteq I$ minimal s.t. $\{a\} \cup \{b\}$ generates A. Indeed $|\{a\}| \geq 1$ and $\{a\} \cup \{b\}$ is a minimal generator of cardinal strictly greater than 1.

5. Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of A-modules. Prove that if M' and M'' are finitely generated, then M is finitely generated.

Let $f : M' \to M$ and $g : M \to M''$ the functions of the short exact sequence. Let's proof that if $\{m'_i\}$ is a finite set of generators of M' and $\{m''_i\}$ is a finite set of generators of M'' then the finite set $\{f(m'_i)\} \cup \{g^{-1}(m''_i)\}$ is a finite set of generators of M. Note that $g^{-1}(x)$ denotes a single element s.t. $g(g^{-1}(x)) = x$, and there always exists one since g is exhaustive.

Let $m \in M$. Then $g(m) = \sum a_i m''_i$, so $m = \sum a_i g^{-1}(m''_i) + k$ with $k \in \ker g$. But indeed $\ker g = \operatorname{Im} f$ so $k = \sum b_i f(m'_i)$. Thus we can express any element with the generators that we have taken and we are finished.

6. Prove that $\mathbb{Z}[\sqrt{d}]$ is a Noetherian ring.

$\mathbb{Z}[\sqrt{d}]$ is a finitely generated $\mathbb{Z}$-module and $\mathbb{Z}$ is Noetherian, thus $\mathbb{Z}[\sqrt{d}]$ is Noetherian.

7. Prove that the ring $\mathbb{Z}[2T, 2T^2, 2T^3, ...] \subset \mathbb{Z}[T]$ is not Noetherian.

Consider the chain $\{2z^i\}_{i \geq 1}$. We claim that the chain does not stabilize. If it stabilized we would have $2z^j = \sum_{i=1}^{j-1} a_i 2z^i$, we need at least one term on the right side with exponent z^j. If one $a_i 2z_i$ has degree j we need that a_i has degree $j - i > 0$, then it is of the form $b2z^{j-i}$ so $4|a_i 2z_i$. Then in the right side we have that $4|c_i$ if c_i is the coefficient of z_i and $4 \not| 2$ the coefficient in the left side, so it does not stabilize.

8. Let M be an A-module and let N_1, N_2 be submodules of M. Prove that if M/N_1 and M/N_2 are Noetherian (Artinian) then $M/(N_1 \cap N_2)$ is Noetherian (Artinian) as well.

To start we need to apply the second theorem of isomorphism (that tells us that $(S+T)/S \simeq S/(S \cap T)$) to $N_1/(N_1 \cap N_2)$. It tells us that $N_1/(N_1 \cap N_2) \simeq (N_1 + N_2)/(N_2) \subseteq M/N_2$. Notice that a submodule of

an Artinian (Noetherian) is Artinian (Noetherian). Since M/N_2 is Artinian (Noetherian), $N_1/(N_1 \cap N_2)$ is Artinian (Noetherian).
Now let's consider the following short exact sequence. $0 \to N_1/(N_1 \cap N_2) \to M/(N_1 \cap N_2) \to M/N_1 \to 0$. Call the morphisms f, g and define $f(\overline{m}) = \overline{m}$ (the inclusion) and $g(\overline{m}) = \overline{m}$. Note that the two quotients of the definition of g are different. Let's check then that the morphism is well defined. We only need that $m \in N_1 \cap N_2$ implies that $g(m) = m \in N_1$, but it's immediate since m is in the intersection.
Let's now check that this is indeed an exact short sequence. f is injective since is an inclusion. In the other hand g is exhaustive since for all $\overline{m} \in M/N_1$ we have that $g(\overline{m}) = m$. Finally we have to check that $\operatorname{Im} f = \ker g$. The inclusion $\operatorname{Im} f \subseteq \ker g$ is shown considering a $\overline{m} \in N_1/(N_1 \cap N_2)$, that implies $m \in N_1$. Then $g \circ f(\overline{m}) = \overline{m}$ but since $m \in N_1$, $\overline{m} = \overline{0}$. In the other hand $g(\overline{m}) = \overline{m} = 0$ implies $m \in N_1$ so $f(\overline{m}) = \overline{m}$ and we have that $\ker f \subseteq \operatorname{Im} g$.
We showed that $N_1/(N_1 \cap N_2)$ is Noetherian (Artinian) and by hypothesis M/N_1 also is. Those two modules form an exact short sequence with $M/(N_1 \cap N_2)$ so this last one has to be Noetherian (Artinian) and thus we are finished.

9. Let M be an A-module, $f : M \to M$ an A-endomorphism. Prove:
(i) If M is Noetherian and f surjective then f is an isomorphism.
We have that $\ker f \subseteq \ker f^2 \subseteq ...$ is an ascending chain. Given that M is Noetherian there exists n s.t. $\ker f^n = \ker f^m$ for all $m \geq n$. Here we have that $\ker f^n \cap \operatorname{Im} f^n = 0$. Let's see this. If $m \in \operatorname{Im} f^n \implies \exists r$ s.t $f^n(r) = m$. Now if $m \in \ker f^n \implies f^n(m) = 0 = f^{2n}(r) \implies f^n(r) = 0 \implies m = 0$. Now as f is surjective we have that $\operatorname{Im} f^n = M \implies \ker f^n = 0 \implies \ker f = 0 \implies f$ is injective and in an abelian category (as the modules) injective and surjective implies isomorphism.
(ii) If M is Artinian and f injective then f is an isomorphism.
We have that $\operatorname{Im} f \supseteq \operatorname{Im} f^2 \supseteq ...$ is a descending chain. M is Artinian so exists an n s.t. $\operatorname{Im} f^n = \operatorname{Im} f^{n+1}$. Now for all elements $m \in M$ we have $f^n(m) = r$ and exists an m' s.t. $f^{n+1}(m') = r$ (since the image of f^n and f^{n+1} is the same). Hence $f^n(m) = f^n(f(m'))$ and by injectivity we have that $f(m') = m$. Then f is exhaustive and since the modules is an abelian category f is an isomorphism.

10. Compute $\operatorname{Hom}_{\mathbb{Q}}(\mathbb{Q}, \mathbb{Z}), \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Q}), \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}/(m), \mathbb{Q})$.
Let's show first that $\operatorname{Hom}_{\mathbb{Q}}(\mathbb{Q}, \mathbb{Z}) = 0$, in other words there exists no homomorphism between $\mathbb{Q}$ and $\mathbb{Z}$ different than 0. Suppose that exists f s.t. $f(a/b) = c \neq 0$, with $a, b, c \in \mathbb{Z}$. Then $2cf(a/(2cb)) = f(2ac/(2cb) = f(a/b) = c$, so $f(a/(2cb)) = 1/2$ that does not lie in $\mathbb{Z}$, so the unique f that can exists is 0.
In the other hand there exists other homomorphisms of $\mathbb{Q}$ into itself. But all of those are determined by the image of 1. Let $f(1) = a \in \mathbb{Q}$. Then we have that for all $b \in \mathbb{Z}$, $bf(1/b) = f(b/b)) = f(1) = a$ so $f(1/b) = a/b$. Furthermore $f(c/b) = f(1/b) + ... + f(1/b)$ c times, then $f(c/b) = ac/b$. In other word $f(x) = ax$. Then we have so many homomorphism as different images that $f(1)$ can have. So we have a homomorphism for each rational and if we denote f_q the homomorphism that satfisfies $f_q(1) = q$ we have that $af_q = f_{aq}$ and $f_q + f_r = f_{q+r}$. So we have that $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Q}) \simeq \mathbb{Q}$.
Again the only homomorphism that can exist between $\mathbb{Z}/(m)$ and $\mathbb{Q}$ is the 0 application. Suppose that $f(\overline{a}) = b$. Then $f(\overline{am}) = f(m\overline{a}) = f(\overline{a}) + ... + \overline{a}) = f(\overline{a}) + ... + f(\overline{a}) = mf(\overline{a}) = bm$. In the other hand $\overline{am} = \overline{0}$ so $f(\overline{am}) = f(\overline{0}) = 0$. Then $0 = bm$ and we need that $b = 0$. This proves that $f = 0$ so $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}/(m), \mathbb{Q}) = 0$.

11. Let A be a ring, M an A-module and $I \subseteq A$ an ideal. Prove $M/IM \simeq A/I \otimes_A M$.
Let's consider first the exact short sequence $0 \to I \overset{f}{\hookrightarrow} A \overset{g}{\to} A/I \to 0$. Where $f(a) = a$ and $g(a) = \overline{a}$. Indeed we have that $\ker f = 0$, $\operatorname{Im}\{f\} = I = \ker g$ and $\operatorname{Im}\{g\} = A/I$ so it' a exact short sequence.

Remember now that $\cdot \otimes_A M$ is a right exact functor. Hence we have: $I \otimes_A M \overset{f'}{\hookrightarrow} A \otimes_A M \overset{g'}{\to} A/I \otimes_A M \to 0$. Where now $f'(a \otimes_A b) = a \otimes_A b$ and $g'(a \otimes_A b) = \overline{a} \otimes_A b$. Hence by the first theorem of isomorphism we have that $A/I \otimes_A M \simeq A \otimes_A M / \ker g'$. Notice that $\ker g' = \operatorname{Im} f' = I \otimes_A M$.

Now we note that every element $a \otimes_A m$ of $A \otimes_A M$ can be written as $1 \otimes_A am$. Using this we can define a morphism $h : (A \otimes_A M)/(I \otimes_A M) \to M/IM$, $h(a \otimes_A m) = am$. We can see that it has been well defined. Indeed if we have an element in $a \otimes m \in I \otimes_A M$ we write it as $1 \otimes_A am$ so $h(1 \otimes_A am) = \overline{am} = \overline{0}$. Note that $h(1 \otimes_A m + 1 \otimes_A m') = h(1 \otimes_A (m + m')) = m + m' = h(1 \otimes_A m) + h(1 \otimes_A m')$. So it's a morphism. More than that it's exhaustive and injective. Hence $A/I \otimes_A M \simeq (A \otimes_A M)/(I \otimes_A M) \simeq M/IM$ as we wanted to see.

12. Let A be a ring and $I, J \subseteq A$ ideals. Prove $A/I \otimes_A A/J \simeq A/(I + J)$.

From `11`: $A/I \otimes_A A/J \simeq (A/J)/(I(A/J))$. Now define a homomorphism $f : A/J \to A/(I + J)$, $f(\overline{x}) = \overline{x}$. To be well defined we need to check that $f(\overline{0}) = \overline{0} \iff (x \in J \implies x \in I + J)$ what is obviously true. Then we can try to apply the first theorem of isomorphism. The $\ker f = \{\overline{x} | f(\overline{x}) = \overline{0}\} \iff \{\overline{x} | x \in I + J\}$. Notice that the $x = i + j, i \in I, j \in J$. Let's define $x' = x - j$. We have that $\overline{x'} = \overline{x}$. More than that $x' = i \in I$. From this $\ker f = I(A/J)$. Then we are finished.

13. Let A be a ring, M, N finitely generated A-modules. Prove:
(i) $M \otimes_A N$ is a finitely generated A-module.
Let $\{m_i\}_I$ a finite number of generators of M and $\{n_j\}_J$ a finite number of generators of N. Then $\{m_i, n_j\}_{I \times J}$ is finite and generates $M \otimes_A N$. Given $m \otimes_A n \in M \otimes_A N$ then $m = \sum a_i m_i$, $n = \sum b_j n_j$. Then $m \otimes_A n = (\sum a_i m_i) \otimes_A (\sum b_j n_j) = \sum (a_i b_j m_i \otimes_A n_j)$.
(ii) If A is Noetherian, then $\operatorname{Hom}_A(M, N)$ is a finitely generated A-module.
Start noting that if $\{m_1, ..., m_r\}$ is a system of generators of M a homomorphism f from M to N is uniquely determined by $\{f(m_1), ..., f(m_r)\}$. Indeed if $m = \sum a_i m_i$ then $f(m) = \sum a_i f(m_i)$. Then we have a homomorphism from $h : \operatorname{Hom}_A(M, N) \to N^r$. Given $f \in \operatorname{Hom}_A(M, N)$, $h(f) = (f(m_1), ..., f(m_r))$. Indeed $ah(f) = a(f(m_1), ..., f(m_r)) = (af(m_1), ..., af(m_r)) = h(af)$ and $h(f + g) = ((f + g)(m_1), ..., (f + g)(m_r)) = (f(m_1), ..., f(m_r)) + (g(m_1), ..., g(m_r)) = h(f) + h(g)$, so it's a homomorphism. The fact that those images uniquely determine f gives that h is injective. Thus by the first theorem of isomorphism we have that $\operatorname{Hom}_A(M, N) \simeq h(\operatorname{Hom}_A(M, N)) \subseteq N^r$. So if we had that any submodule of N^r was finitely generated we would be finished.
We will see this seeing that N^r is in fact Noetherian. We have that N is a finitely generated module over A, a Noetherian ring, so it's also Noetherian. Now N^r is also finitely generated over A so it is in fact Noetherian. Since every submodule of a Noetherian module is finitely generated and $\operatorname{Hom}_A(M, N)$ is a submodule $\operatorname{Hom}_A(M, N)$ is finitely generated.

14. Let A be a local ring, M, N finitely generated A-modules. Prove that $M \otimes_A N = 0$ if and only if $M = 0$ or $N = 0$. Prove that the result is no longer true if the ring is not local.
$\Longleftarrow$ needs no further explanation.
$\Longrightarrow$ let's use the same idea that we used in **3**. Let m the maximal ideal of A-. Define $f : M \otimes_A N \to (M/\mathrm{m}M) \otimes_{(A/\mathrm{m})} (N/\mathrm{m}N)$, $f(a \otimes_A b) = \overline{a} \otimes_{(A/\mathrm{m})} \overline{b})$. We saw in **3.** that changing the A for A/m once we took modulo was okay. Now we have that $(M/\mathrm{m}M) \otimes_{(A/\mathrm{m})} (N/\mathrm{m}N) = 0$. Notice that now the components of the tensor product are finite dimension vector spaces. If the tensor product equals 0 one of the components has to equal 0. Without lose of generality we have $M/\mathrm{m}M = 0$. We are in position to use Nakayama's lemma to say that we need $M = 0$ and thus we are finished.
For the seeing that the locality of the ring is needed take A with two different maximal ideals I and J. Consider $A/I \otimes_A A/J$. Indeed those are not 0. Now consider any element $\overline{a} \otimes_A \overline{b} \in A/I \otimes_A A/J$

and $i \in I, i \notin J$. Then for being A/J a field we have a i' s.t. $\overline{ii'} = \overline{1}$ in A/J. So $\overline{a} \otimes_A \overline{b} = \overline{a} \otimes_A \overline{ii'b} = \overline{ia} \otimes_A \overline{i'b} = \overline{0} \otimes_A \overline{i'b} = 0$. Using that $i \in I \implies ai \in I$. Thus any element is equal to 0 so is the space.

15. Let M be a finitely generated A-module and let $S \subseteq A$ be a multiplicatively closed set. Prove that $S^{-1}M = 0$ if and only if there exists $s \in S$ such that $sM = 0$.
$S^{-1}M = 0 \iff \forall m \in M, s \in S, (m/s \equiv 0/1 \iff \exists s' \in S \text{ s.t. } s'(m) = 0) \iff \forall m \in M \exists s \in S$ s.t. $ms = 0$.
Then for $\Longleftarrow$ we just have to take the s s.t. $sM = 0$ for all m.
For the reverse implication let $\{m_i\}$ be finite number of generators of M. Then by hypothesis for each m_i exists $s_i \in S$ s.t. $m_i s_i = 0$. Consider $s = \prod s_i$. Since S in multiplicatively closed $s \in S$. Know all $m \in M$ can me expressed $m = \sum a_i m_i$. Hence $sm = s \sum a_i m_i = \sum a_i s m_i = \sum b_i s_i m_i = \sum 0 = 0$ so we are finished.

16. Let $S \subseteq A$ be a multiplicatively closed set. Prove that the localization functor S^{-1}- is exact.
Remember first that $S^{-1}(M) = \{\frac{m}{s} | m \in M, s \in S\}$, with the relation $\frac{m}{a} \sim \frac{m'}{a'} \iff \exists t \in S$ s.t. $t(ma' - m'a) = 0$. Also if $f : M \to N$, $S^{-1}(f) : S^{-1}M \to S^{-1}N$, $S^{-1}(f)(\frac{m}{a}) = \frac{f(m)}{b}$. This is well defined.
Let's now see that if we have a exact short sequence $0 \to M' \to M \to M'' \to 0$, with functions f and g then $S^{-1}M' \to S^{-1}M \to S^{-1}M''$, with functions $f' := S^{-1}(f)'$ and $g' := S^{-1}(g)$ is also exact.
Firstly we can see that exhaustivity and injectivity are preserved.
The $\ker f' = \{\frac{m}{a} | f'(\frac{m}{a}) = \frac{f(m)}{a} = 0\}$. But $\frac{f(m)}{a} = 0 \iff \exists t \in S$ s.t. $t(f(m) - a \cdot 0) = tf(m) = f(tm) = 0$. By injectivity of f then $tm = 0$ so $\frac{m}{a} = 0$ thus $\ker f' = 0$ and f' is injective.
To see that g' is exhaustive we have to find for all $\frac{m}{a} \in M''$ a $\frac{m'}{a'} \in M$ s.t. $g'(\frac{m'}{a'}) = \frac{m}{a}$. But $g'(\frac{m'}{a'}) = \frac{g(m')}{a'}$. By exhaustivity of g we know that we can pick a m' s.t. $g(m') = m$ and choosing $a' = a$ we have that $g'(\frac{m'}{a}) = \frac{m}{a}$ thus we are finished.
To end we have to check that $\text{Im}\{f'\} = \ker g'$, knowing that $\text{Im}\{f\} = \ker g$. Let's see that $\text{Im} f' \subseteq \ker g' \iff g' \circ f' = 0$, $g' \circ f'(\frac{m}{a}) = g'(\frac{f(m)}{a}) = \frac{g(f(m))}{a}$, by hypothesis $g \circ f = 0$ so $\frac{g(f(m))}{a} = 0$ and we are finished.
We are only left to check that $\ker g' \subseteq \text{Im} f'$. $\frac{m}{a} \in \ker g' \iff g'(\frac{m}{a}) = \frac{g(m)}{a} = 0 \iff \exists t \in S$ s.t. $tg(m) = g(tm) = 0$. By the same condition between f and g there exists $m' \in M'$ s.t. $f(m') = tm$. Hence $f'(\frac{m'}{at}) = \frac{tm}{at} = \frac{m}{a}$. Note that we used that S is closed under multiplication and the $at \in S$ because $a, t \in S$. Then $\ker g' \subseteq \text{Im} f'$ and $\ker g' = \text{Im} f'$ and we proved that the sequence under the localization functor is still exact so the localization functor is exact.

17. Let M be an A-module. We say that it is simple if it doesn't contain any non-trivial submodule (i.e. if $N \subseteq M$ is a submodule, then $N = 0$ or $N = M$). Prove:
(i) Every simple module is cyclic.
Consider a non-zero element m of M. Take the elements generated by it (m). It's a submodule and it's not empty, then $(m) = M$ and M is generated by m, so M is cyclic.
(ii) If M, N are simple A-modules and $f : M \to N$ is an homomomorphism, then $f = 0$ or f is an isomorphism. Since the modules are an abelian cathegory for f being an isomorphism we only need to check exhaustivity and injectivity. Consider first $\ker f$. Since $\ker f$ is a submodule of M we have two cases. If $\ker f = M$ then $f = 0$.
Assume the opposite, so $\ker f = 0$ and f is injective. Now we have to check exhaustivity. We can not say directly that $\text{Im} f$ is a submodule on N since it is false in general.
Take a non-zero element of M, call it m. Now since $\ker f = 0$ we know that $f(m)$ is non-zero. By **(i)** we know that $(f(m)) = N$. Indeed we also know that $(m) = M$. We just have to check that

$f((m)) = (f(m))$. An element of (m) is of the form am with $a \in A$. An element of $(f(m))$ is of the form $af(m)$ with $a \in A$. But indeed $af(m) = f(am) \in f((am))$ hence $f(M) = f((m)) = (f(m)) = N$ and we are finished.

18. Let A be an integral domain and let M be an A-module. We say that $m \in M$ is a torsion element if there exists $a \in A$
$\{0\}$ such that $am = 0$. Let $T(M)$ be the set of torsion elements. Prove:
(i) $T(M)$ is a submodule of M
We have to check that $T(M)$ is closed under addition and multiplication by scalar. If we have $m, n \in M$ s.t $\exists a, b \neq 0 \in A$ with $am = bn = 0$ we have that $ab(m+n) = b(am) + a(bn) = b \cdot 0 + a \cdot 0 = 0 + 0 = 0$. Since A is a domain $ab \neq 0$. And if $c \in A$ we have that $a(cm) = c(am) = c \cdot 0 = 0$ so it's also closed by multiplication by scalar.
(ii) $M/T(M)$ has no torsion.
Let $\overline{m} \in T(M/T(M))$. It follows that there exists a non-zero $a \in A$ s.t. $a\overline{m} = \overline{am} = \overline{0}$. This implies that $am \in T(m)$. Hence there exists a non-zero $b \in A$ s.t. $b(am) = 0 = (ab)m$. Note that ab is non-zero. Then m is in $T(M)$ so $\overline{m} = 0$ and the torsion of $M/T(M)$ is 0.
(iii) If $f : M \to N$ is A-linear, then $f(T(M)) \subseteq T(N)$.
Let $m \in T(M)$, so there exists a non-zero $a \in A$ s.t. $am = 0$. Then we have that $af(m) = f(am) = 0$ so $f(m) \in T(N)$ and we are finished.
(iv) If $0 \to M' \to M \to M'' \to 0$ is an exact sequence then $0 \to T(M') \to T(M) \to T(M'')$ is exact.
Let's call f the function between M' and M and g the one between M and M''. Let f' and g' the restrictions of those functions to the torsions. From **iii** we know that are well defined.
The injectivity of f' follows from the injectivity of f. Then we only need to see that $\ker g' = \operatorname{Im} f'$.
The inclusion $\operatorname{Im} f' \subseteq \ker g'$ follows from the same condition between f and g.
We have to check now that $\ker g' \subseteq \operatorname{Im} f'$. Let $m \in T(M)$ s.t. $g'(m) = 0 = g(m)$. We know from the short sequence condition between f and g that exists an $n \in M'$ s.t. $f(n) = m$. We are left to show that this n lives in $T(M')$. Since m is in $T(M)$ there exists a $a \in A$ s.t. $am = 0$. Hence $af(n) = f(an) = am = 0$. It follows that $an \in \ker f$ but $\ker f = 0$ so $an = 0$ and $n \in T(M')$ as we wanted.